TOPIC: Matrices

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39 Matrices

39.1 Matrices: Addition and Subtraction

The numbers shown in a matrix can be in many different shapes and sizes; for example,

(a) $\begin{pmatrix} 2 & -1 \\ 4 & \frac{3}{2} \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 & 5 \\ \frac{1}{2} & 4 \\ 3 & -1 \end{pmatrix}$

We describe these shapes and sizes by giving the **dimension** of the matrix. The dimension is

(number of *rows* in the matrix) by (number of *columns*)

(Rows are along the horizontal; columns are vertical.)

Worked Example 1

What is the dimension of each of the matrices shown above?

Solution

(a) 2 by 2 (as it has 2 rows and 2 columns)

(b) 1 by 3

(c) 2 by 1

(d) 3 by 2

In general, if a matrix has m rows and n columns, its dimension is

m by n

We often write this as

$m \times n$

but the '×' sign does *not* mean multiply; you read this as "m by n".

You can *add* and *subtract* matrices by adding and subtracting their corresponding **elements**. Matrices have to have the same dimensions in order to be added or subtracted.

Worked Example 2

Calculate

(a) $\begin{pmatrix} 5 & -1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ -2 & 5 \end{pmatrix}$

(b) $\begin{pmatrix} 4 & 3 & 1 \\ 6 & -1 & 4 \end{pmatrix} - \begin{pmatrix} 10 & 2 & -5 \\ 4 & 5 & -8 \end{pmatrix}$

Solution

(a) $\begin{pmatrix} 5 & -1 \\ 2 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 1 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 5+5 & -1+1 \\ 2+(-2) & 0+5 \end{pmatrix}$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$$

(b) $\begin{pmatrix} 4 & 3 & 1 \\ 6 & -1 & 4 \end{pmatrix} - \begin{pmatrix} 10 & 2 & -5 \\ 4 & 5 & -8 \end{pmatrix} = \begin{pmatrix} 4-10 & 3-2 & 1-(-5) \\ 6-4 & -1-5 & 4-(-8) \end{pmatrix}$

$$= \begin{pmatrix} -6 & 1 & 6 \\ 2 & -6 & 12 \end{pmatrix}$$

Matrices are equal if they have the same dimensions and corresponding elements are equal.

Worked Example 3

Given that $\begin{pmatrix} a & 3 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 4 & c \\ d & -5 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, find the values of *a, b, c* and *d*.

Solution

$$\begin{pmatrix} a & 3 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 4 & c \\ d & -5 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

That is,

$$\begin{pmatrix} a-4 & 3-c \\ -1-d & b+5 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

As the matrices are equal, each element must be equal; that is

$$\begin{aligned} a-4 &= 5 &\Rightarrow\quad & a = 9 \\ 3-c &= 0 &\Rightarrow\quad & c = 3 \\ -1-d &= 0 &\Rightarrow\quad & d = -1 \\ b+5 &= 5 &\Rightarrow\quad & b = 0 \end{aligned}$$

Exercises

1. What is the dimension of each of the following matrices?

(a) $\begin{pmatrix} 4 & -1 \\ 2 & 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}$ (c) $\begin{pmatrix} -1 & 2 & 3 \\ 0 & 1 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 1 \end{pmatrix}$ (e) $\begin{pmatrix} 3 & -6 & -1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

2. For the matrices,

$$A = \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix} \qquad C = \begin{pmatrix} 7 & 0 \\ 0 & 5 \end{pmatrix}$$

find

(a) $A + B$

(b) $A - B$

(c) $A + B - C$

(d) $C - A - B$

3. For the matrices

$$A = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix} \qquad E = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad F = \begin{pmatrix} 5 & -1 & -2 \end{pmatrix}$$

find, where possible,

(a) $A + B$

(b) $E - A$

(c) $F - D - C$

(d) $B + C$

(e) $F - (D - C)$

(f) $A - F$

(g) $C - (F + D)$

4. Given that

$$\begin{pmatrix} a & 5 \\ 2 & b \end{pmatrix} + \begin{pmatrix} 4 & c \\ d & -1 \end{pmatrix} = \begin{pmatrix} 0 & 10 \\ 4 & 0 \end{pmatrix}$$

find the value of the constants a, b, c and d.

5. Given that

$$\begin{pmatrix} -2 & 5 & 1 \\ 3 & -1 & -4 \end{pmatrix} + \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 4 & 0 & -1 \\ 7 & 3 & 0 \end{pmatrix}$$

find the value of the constants a, b, c, d, e and f.

39.2 Matrices: Multiplication

There are two types of multiplication defined for matrices.

(A) SCALAR multiplication

Here each element of the matrix is multiplied by a scalar (number).

For example,

$$2\begin{pmatrix} 4 & 3 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2\times 4 & 2\times 3 \\ 2\times 1 & 2\times(-1) \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ 2 & -2 \end{pmatrix}$$

Worked Example 1

If

$$A = \begin{pmatrix} -5 & 2 \\ 6 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

what are

(a) $2A$ (b) $\frac{1}{2}A$ (c) $(-4)B$?

Solution

(a) $$2A = 2\begin{pmatrix} -5 & 2 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} 2\times(-5) & 2\times 2 \\ 2\times 6 & 2\times 0 \end{pmatrix} = \begin{pmatrix} -10 & 4 \\ 12 & 0 \end{pmatrix}$$

(b) $$\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} -5 & 2 \\ 6 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\times(-5) & \frac{1}{2}\times 2 \\ \frac{1}{2}\times 6 & \frac{1}{2}\times 0 \end{pmatrix} = \begin{pmatrix} \frac{-5}{2} & 1 \\ 3 & 0 \end{pmatrix}$$

(c) $$(-4)B = (-4)\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} (-4)\times 1 & (-4)\times(-1) & (-4)\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 4 & -4 \end{pmatrix}$$

(B) MATRIX multiplication

You can multiply two matrices, A and B, together and write

$$C = AB \quad (\text{or } A \times B)$$

only if the number of columns of $A =$ number of rows of B; that is, A has dimension $m \times n$, B has dimension $n \times k$. The resulting matrix, C, has dimensions $m \times k$.

To find C, we multiply corresponding elements of each row of A by elements of each column of B and add. The following examples show you how the calculation is done.

Worked Example 2

If $A = \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, find AB.

Solution

First check the dimension: A is 2×2 and B is 2×1, so $C = AB$ is defined. (The number of columns of $A =$ the number of rows of B) and C is a 2×1 matrix:

$$\begin{aligned} C &= \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times (-1) + (-1) \times 2 \\ (-2) \times (-1) + 3 \times 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 - 2 \\ 2 + 6 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 8 \end{pmatrix} \end{aligned}$$

Worked Example 3

If $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ 5 & 0 \end{pmatrix}$, calculate AB and BA.

Solution

A is 2×2 and B is 2×2, so AB is defined and

$$\begin{aligned} AB &= \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 5 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 \times (-1) + 1 \times 5 & 2 \times 2 + 1 \times 0 \\ 0 \times (-1) + (-1) \times 5 & 0 \times 2 + (-1) \times 0 \end{pmatrix} \\ &= \begin{pmatrix} -2 + 5 & 4 + 0 \\ 0 - 5 & 0 + 0 \end{pmatrix} \\ &= \begin{pmatrix} -3 & 4 \\ -5 & 0 \end{pmatrix} \end{aligned}$$

Similarly, BA is defined and

$$BA = \begin{pmatrix} -1 & 2 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} (-1) \times 2 + 2 \times 0 & (-1) \times 1 + 2 \times -1 \\ 5 \times 2 + 0 \times 0 & 5 \times 1 + 0 \times (-1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 + 0 & -1 - 2 \\ 10 + 0 & 5 + 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -3 \\ 10 & 5 \end{pmatrix}$$

Note

$AB \neq BA$ and this shows that matrix multiplication is *not* commutative (that is, the calculation cannot be reversed with the same result).

Worked Example 4

Given that $A = \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

determine whether or not the following multiplications are defined, and, if they are, find the resulting matrix.

(a) AB (b) BC (c) AC (d) CA (e) BCA

Solution

(a) AB is *not* defined, as A is 1×3 and B is 1×2, and $3 \neq 1$.

(b) BC *is* defined. As B is 1×2 and C is 2×1, BC is 1×1.

That is

$$BC = \begin{pmatrix} 2 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \times 2 - 2 \times 4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 - 8 \end{pmatrix}$$

$$= \begin{pmatrix} -4 \end{pmatrix}$$

(c) AC is *not* defined, as A is 1×3 and C is 2×1.

(d) CA *is* defined; C is 2×1 and A is 1×3 so CA is 2×3.

That is

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 \times 1 & 2 \times 0 & 2 \times (-1) \\ 4 \times 1 & 4 \times 0 & 4 \times (-1) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & -2 \\ 4 & 0 & -4 \end{pmatrix}$$

(e) We already know that BC is a 1×1 matrix and A is 1×3, so BCA *is* defined and is 1×3.

That is

$$BCA = (-4)\begin{pmatrix} 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -4 & 0 & 4 \end{pmatrix}$$

Worked Example 5

Let $A = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$.

Calculate

(a) $A + B$

(b) AB

(c) BA

(d) $A^2 - B$ $\quad (A^2$ means $AA)$

(CXC)

Solution

(a) $$A + B = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ -3 & 5 \end{pmatrix}$$

(b) $$AB = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 \times 3 + 0 \times (-1) & 1 \times (-5) + 0 \times 2 \\ -2 \times 3 + 3 \times (-1) & (-2) \times (-5) + 3 \times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -5 \\ -9 & 16 \end{pmatrix}$$

(c) $$BA = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 3 \times 1 + (-5) \times (-2) & 3 \times 0 + (-5) \times 3 \\ -1 \times 1 + 2 \times (-2) & -1 \times 0 + 2 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & -15 \\ -5 & 6 \end{pmatrix}$$

(d) $$A^2 - B = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 + 0\times(-2) & 1\times 0 + 0\times 3 \\ -2\times 1 + 3\times(-2) & -2\times 0 + 3\times 3 \end{pmatrix} - \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -8 & 9 \end{pmatrix} - \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 5 \\ -7 & 7 \end{pmatrix}$$

Exercises

1. For the matrices $A = \begin{pmatrix} 2 & 0 \\ 4 & -6 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, find

   (a) $3A$ (b) $\frac{1}{2}A$ (c) $2B$

2. Find the value of k and the value of x so that

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} + k\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 7 \\ x & 0 \end{pmatrix}$$

3. Find the values of a, b, c and d so that

$$2\begin{pmatrix} a & 0 \\ 1 & b \end{pmatrix} - 3\begin{pmatrix} 1 & c \\ d & -1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ -4 & -4 \end{pmatrix}$$

4. Find the values of a, b, c and d so that

$$\begin{pmatrix} 5 & a \\ b & 0 \end{pmatrix} - 2\begin{pmatrix} c & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 \\ 3 & d \end{pmatrix}$$

5. Given the dimensions of the following matrices,

Matrix	A	B	C	D	E
Dimension	2 × 2	1 × 2	1 × 3	3 × 2	2 × 3

   give the dimensions of these matrix products:

   (a) BA (b) DE (c) CD

   (d) ED (e) AE (f) DA

6. Find these products:

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$

7. The matrix $A = \begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$ and the matrix $B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Find:

(a) AB (b) A^2

8. The matrices A, B and C are given by

$$A = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} -3 & -2 \end{pmatrix}$$

Determine whether or not the following products are possible and find the products of those that are.

(a) AB (b) AC (c) BC

(d) BA (e) CA (f) CB

9. Find in terms of a, $\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$.

10. Find in terms of x, $\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$.

11. The matrix $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

Find:

(a) A^2 (b) A^3

39.3 Inverse Matrices: Solving Linear Equations

In this section, we will restrict ourselves to square matrices, that is, matrices in which the number of rows = number of columns $(m = n)$.

For example,

$$A = \begin{pmatrix} 4 & 1 \\ -2 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}, \; C = \begin{pmatrix} 4 & 3 \\ 4 & 3 \end{pmatrix}, \; D = \begin{pmatrix} 5 \end{pmatrix}$$

We will concentrate on 2×2 matrices (examples A and C above). We start by defining the **determinant** of a 2×2 matrix.

If $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its determinant is defined by

$$\det M = ad - cb$$

Note that it is just a number, So, for the examples above,

$$\begin{aligned} \det A &= 4 \times 1 - (-2) \times 1 \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

and

$$\begin{aligned} \det C &= 4 \times 3 - 4 \times 3 \\ &= 12 - 12 \\ &= 0 \end{aligned}$$

For matrix *M,* if $\det M = 0$ we call it **singular** and if $\det M \neq 0$, it is **non-singular**.

Hence *A* is non-singular and *C* is singular.

Worked Example 1

M is the matrix $\begin{pmatrix} -4 & x \\ -10 & -5 \end{pmatrix}$.

Calculate the value of *x* which would make *M* a singular matrix.

(CXC)

Solution

M is singular if $\det M = 0$.

$$\begin{aligned} \det M &= (-4) \times (-5) - x \times (-10) \\ &= 20 + 10x \end{aligned}$$

So $\det M = 0$ when

$$\begin{aligned} 20 + 10x &= 0 \\ 10x &= -20 \\ x &= -2 \end{aligned}$$

For a 2×2 matrix, we will define its **inverse**, denoted by *M,* as the matrix that satisfies

$$XM = MX = I$$

Here *I* is the identity matrix, defined as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and we denote *X* by M^{-1}.

It is not always possible to find an inverse, as we will see in the following result.

Theorem

The inverse of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$M^{-1} = \frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

and only exists if $\det M \neq 0$.

The matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is called the **adjoint** matrix and in denoted by $\operatorname{adj} A$. Note that, on one diagonal the numbers are reversed and on the other they are multiplied by (-1).

Proof

We will show that $M^{-1}M = I$.

Note that

$$\frac{1}{\det M}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\det M}\begin{pmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{pmatrix}$$

$$= \frac{1}{\det M}\begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ad - bc}{\det M} & 0 \\ 0 & \frac{ad - bc}{\det M} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (\text{since } \det M = ad - bc)$$

$$= I$$

So $M^{-1}M = I$ and you can show that $MM^{-1} = I$ in the same way. Thus we can find the inverse of a 2×2 matrix provided that $\det M \neq 0$; that is, it is non-singular.

Worked Example 2

Find the inverse of

$$A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

and show that $AA^{-1} = A^{-1}A = I$.

Solution

Since $\det A = 5 \times 2 - 3 \times 3 = 10 - 9 = 1$, A is non-singular and

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \frac{1}{1}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$$

We now check this by calculating

$$AA^{-1} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and similarly,

$$A^{-1}A = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note

We can use this concept of finding the inverse to solve simultaneous equations.

For example, if

$$5x + 3y = 4$$
$$3x + 2y = 2$$

then this can be written as

$$AX = B$$

when $A = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

Check:

$$AX = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5x + 3y \\ 3x + 2y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = B$$

Multiply $AX = B$ by A^{-1} to give

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$ from the result above, so

$$IX = A^{-1}B$$

But $IX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = X$ giving

$$X = A^{-1}B$$

$$= \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

That is, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

So the solution is $x = 2,\ y = -2$. We will check these values in the original equation:

$$5x + 3y = 5 \times 2 + 3 \times (-2) = 10 - 6 = 4$$

$$3x + 2y = 3 \times 2 + 2 \times (-2) = 6 - 4 = 2$$

and this verifies the solution.

Worked Example 3

Given that $M = \begin{pmatrix} 2 & 5 \\ 7 & 15 \end{pmatrix}$.

(a) Show that M is a non-singular matrix.

(b) Write down the inverse of M.

(c) Write down the 2×2 matrix which is equal to the product of $M \times M^{-1}$.

(d) Pre-multiply both sides of the following matrix equation by M^{-1}

$$\begin{pmatrix} 2 & 5 \\ 7 & 15 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 17 \end{pmatrix}$$

Hence solve for x and y. *(CXC)*

Solution

(a) $\det M = \begin{pmatrix} 2 & 5 \\ 7 & 15 \end{pmatrix}$

$$= 2 \times 15 - 5 \times 7$$

$$= 30 - 35$$

$$= -5$$

So $\det M \neq 0$ and M is non-singular.

(b) $$M^{-1} = \frac{1}{\det M} \times \operatorname{adj} M$$

$$= \frac{1}{(-5)} \begin{pmatrix} 15 & -5 \\ -7 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & 1 \\ \frac{7}{5} & \frac{-2}{5} \end{pmatrix}$$

(c) $M \times M^{-1} = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $M \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 17 \end{pmatrix}$

Multiply by M^{-1} to give

$$M^{-1} M \begin{pmatrix} x \\ y \end{pmatrix} = M^{-1} \begin{pmatrix} -3 \\ 17 \end{pmatrix}$$

$$I\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ \frac{7}{5} & \frac{-2}{5} \end{pmatrix}\begin{pmatrix} -3 \\ 17 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 26 \\ -11 \end{pmatrix}$$

So $x = 26$ and $y = -11$.

Exercises

1. Determine which of these matrices are singular and which are non-singular. For those that are non-singular find the inverse matrix.

   (a) $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$

   (d) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ (e) $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

2. Find the value of a for which these matrices are singular.

   (a) $\begin{pmatrix} a & 1+a \\ 3 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$ (c) $\begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

3. Given that $AB = C$,

   (a) find an expression for B

   (b) find B when $A = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$

4. Use inverse matrices to solve these simultaneous equation.

   (a) $4x - y = -1$
   $-2x + 3y = 8$

   (b) $4x - y = 11$
   $3x + 2y = 0$

   (c) $5x + 2y = 3$
   $3x + 4y = 13$

5. Solve, using matrix methods,

   $3x - 2y = 3$
   $x + 4y = 8$

6. Solve, using matrix methods as far as possible, each of the following sets of simultaneous equations.

   (a) $x + y = 2$, $2x + 3y = 5$ (b) $x + y = 2$, $2x + 2y = 4$ (c) $x + y = 2$, $2x + 2y = 5$

7. (a) Find the matrix P if

$$PQ = QP = I$$

where Q is the matrix $\begin{pmatrix} 2 & 1 \\ 4 & 5 \end{pmatrix}$.

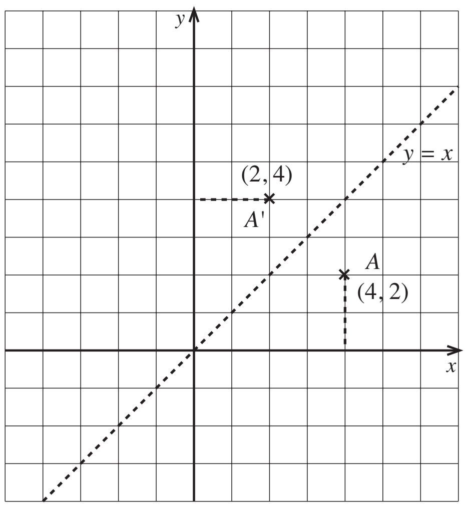
   (b) Using part (a), solve the equation

$$2x + y = -1$$
$$4x + 5y = -11$$

39.4 Geometrical Transformations

In earlier units in this strand we met transformations, including rotations, reflections and enlargements. A convenient way to define these is by using a matrix approach.

For example, consider a **reflection** in the line $x = y$. The point A, (4, 2) is transformed to A' (2, 4) as shown opposite. Similarly, the point P (x, y) is transformed to P' (y, x).



We can write the new coordinates in matrix form as

$$X' = MX \quad \text{or} \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

as multiplying out the matrix on the right hand side gives

$$x' = y$$
$$y' = x$$

that is, the coordinates of P' are (x, y).

So the matrix

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

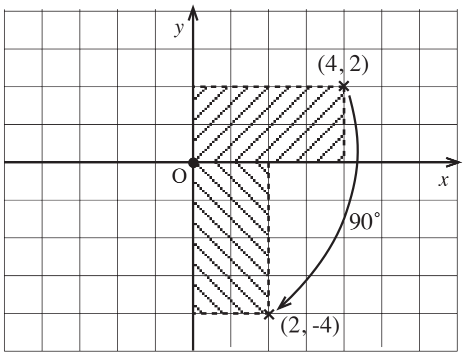
defines the transformation "reflection in the line $y = x$".

Similarly for other reflections; all are summarised below.

Reflection	*Matrix Transformation*
In the line $y = x$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
In the line $y = -x$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$
In the y-axis	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
In the x-axis	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

A similar process is used for **rotations**.

Consider a rotation of $90°$ in the clockwise direction about the origin, O, as shown opposite for the point



$$(4, 2) \rightarrow (2, -4)$$

Similarly,

$$(x, y) \rightarrow (y, -x)$$

for any point of the shape, and in matrix form, $X^1 = MX$, or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} y \\ -x \end{pmatrix}$$

That is, $x' = y$

$y' = -x$

So the coordinates of $P\,(x,\ y)$ become $P'\,(y, -x)$ and for the rotation of the shape shown,

$$(0, 2) \rightarrow (2, 0)$$

$$(4, 0) \rightarrow (0, -4)$$

$$(0, 0) \rightarrow (0, 0)$$

The matrix

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

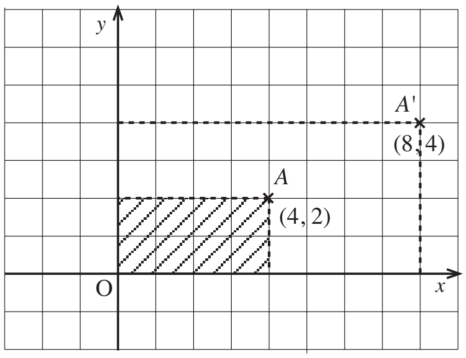
corresponds to a rotation of $90°$ in the clockwise direction.

Similarly for other rotations; all are summarised below.

Rotation	*Matrix Transformation*
$90°$ clockwise	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$90°$ anticlockwise	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$180°$ (clockwise or anticlockwise)	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Enlargements can be treated in a similar way.

For an enlargement of the shape shown, centre O and scale factor 2, the diagram opposite shows that



$$(4, 2) \rightarrow (8, 4)$$

and similarly

$$(x, y) \rightarrow (2x, 2y)$$

For example,

$$(0, 2) \rightarrow (0, 4)$$

$$(4, 0) \rightarrow (8, 0)$$

$$(0, 0) \rightarrow (0, 0)$$

In matrix terms, $X^1 = MX$ when

$$\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

That is, $x^1 = 2x$

$y^1 = 2y$

so that the coordinates of $P(x, y)$ become $P'(2x, 2y)$.

The matrix $M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ represents an enlargement of scale factor 2.

For scale factor *n*, with centre of enlargement being the origin, the matrix will be

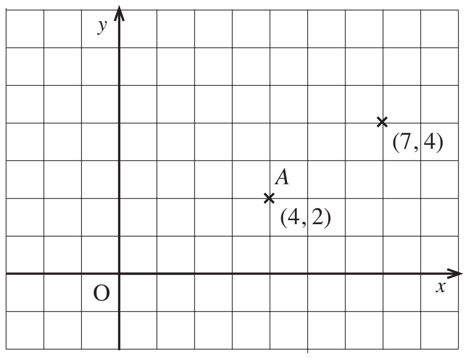
$$M = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$$

For the CXC examination, you do need to know these matrix transformations, as you will see in the Worked Examples.

Note

Points that do not change position after transformation are called **invariant** points. For example, points on the line $y = x$ remain in the same position after reflection in the line $y = x$. These are the invariant points for the transformation "reflective in the line $y = x$".

Finally, we can look at **translations** in a similar way.



If we move $x \to x + 3$ and

$$y \to y + 2$$

then for the point *A* (4, 2),

$$A' \text{ is } (7, 4)$$

and $P(x, y)$ is translated

to $P'(x', y')$ where

$$x' = x + 3$$

$$y' = y + 2$$

In matrix form, we have

$$X' = M + X \text{ when } X' = \begin{pmatrix} x' \\ y' \end{pmatrix}, \; M = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \end{pmatrix}$$

In general, for the translation $x \to x + a, \; y \to y + b$,

then

$$X' = M + X \quad \text{where} \quad M = \begin{pmatrix} a \\ b \end{pmatrix}$$

Worked Example 1

The matrix, K, maps the point $S(1, 4)$ onto $S'(-4, -1)$ and the point $T(3, 5)$ onto $T'(-5, -3)$. Given that $K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

(a) Express as a matrix equation, the relationship between

(i) K, S and S'

(ii) K, T and T'.

(b) Hence, determine the values of *a, b, c* and *d*.

(c) Describe COMPLETELY the geometric transformation which is represented by the matrix K.

(CXC)

Solution

(a) (i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$

(ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

(b) $a + 4b = -4$ ①

$c + 4d = -1$ ②

$3a + 5b = -5$ ③

$3c + 5d = -3$ ④

We can solve for *a* and *b* from ① and ③:

Multiply ① by 3 to give

$$3a + 12b = -12$$

Taking ③ from this gives $7b = -12 + 5 = -7 \Rightarrow b = -1$.

From ①,

$$a + 4 \times (-1) = -4 \Rightarrow a = 0$$

and $a = 0, b = -1$ satisfies ③.

Multiply ② by 3 to give

$$3c + 12d = -3$$

Taking ④ from this equation gives

$$7d = 0 \Rightarrow d = 0$$

and substituting in ② gives $c = -1$.

Hence,

$$K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(c) This is a *reflection* in the line $y = -x$.

Worked Example 2

A triangle XYZ, with coordinates $X\,(4, 5)$, $Y\,(-3, 2)$ and $Z\,(-1, 4)$ is mapped onto triangle $X'\,Y'\,Z'$ by a transformation $M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) Calculate the coordinates of the vertices of triangle $X'\,Y'\,Z'$.

(b) A matrix $N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ maps triangle $X'\,Y'\,Z'$ onto triangle $X''\,Y''\,Z'''$.

Determine the 2×2 matrix, Q, which maps triangle XYZ onto $X''\,Y''\,Z''$.

(c) Show that the matrix which maps triangle $X''\,Y''\,Z''$ back onto triangle XYZ is equal to Q.

(CXC)

Solution

(a) $X' = MX = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ i.e. $X'\,(-4, 5)$

$Y' = MY = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ i.e. $Y'\,(3, 2)$

$Z' = MZ = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ i.e. $Z'\,(1, 4)$

(b) $X'' = NX' = NMX$, so $X'' = QX$ where

$$Q = NM = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(c) Check:

$$QX'' = Q^2 X \text{ and } Q^2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $QX'' = X$ and similarly for Y'' and Z''.

Worked Example 3

(a) Write down the matrix

(i) M_y that represents reflection in the y-axis

(ii) R_l that represents a rotation of $180°$ about the origin.

(b) Determine the single matrix, U, that represents a transformation, M_y, followed by another transformation, R_l.

(c) Describe geometrically the transformation represented by

(i) $R_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(ii) $E = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(d) On graph paper, using a scale of 1 cm to represent 1 unit on each axis, draw the pentagon *ABCDE* with vertices *A* (1, 2), *B* (4, 2), *C* (4, 5), *D* (2, 6) and *E* (1, 5).

(e) Draw the image of *ABCDE* under the transformation represented by

(i) R_p, and label that image $A' B' C' D' E'$

(ii) E, and label that image $A'' B'' C'' D'' E''$.

(CXC)

Solution

(a) (i) $M_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(ii) $R_l = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $U = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

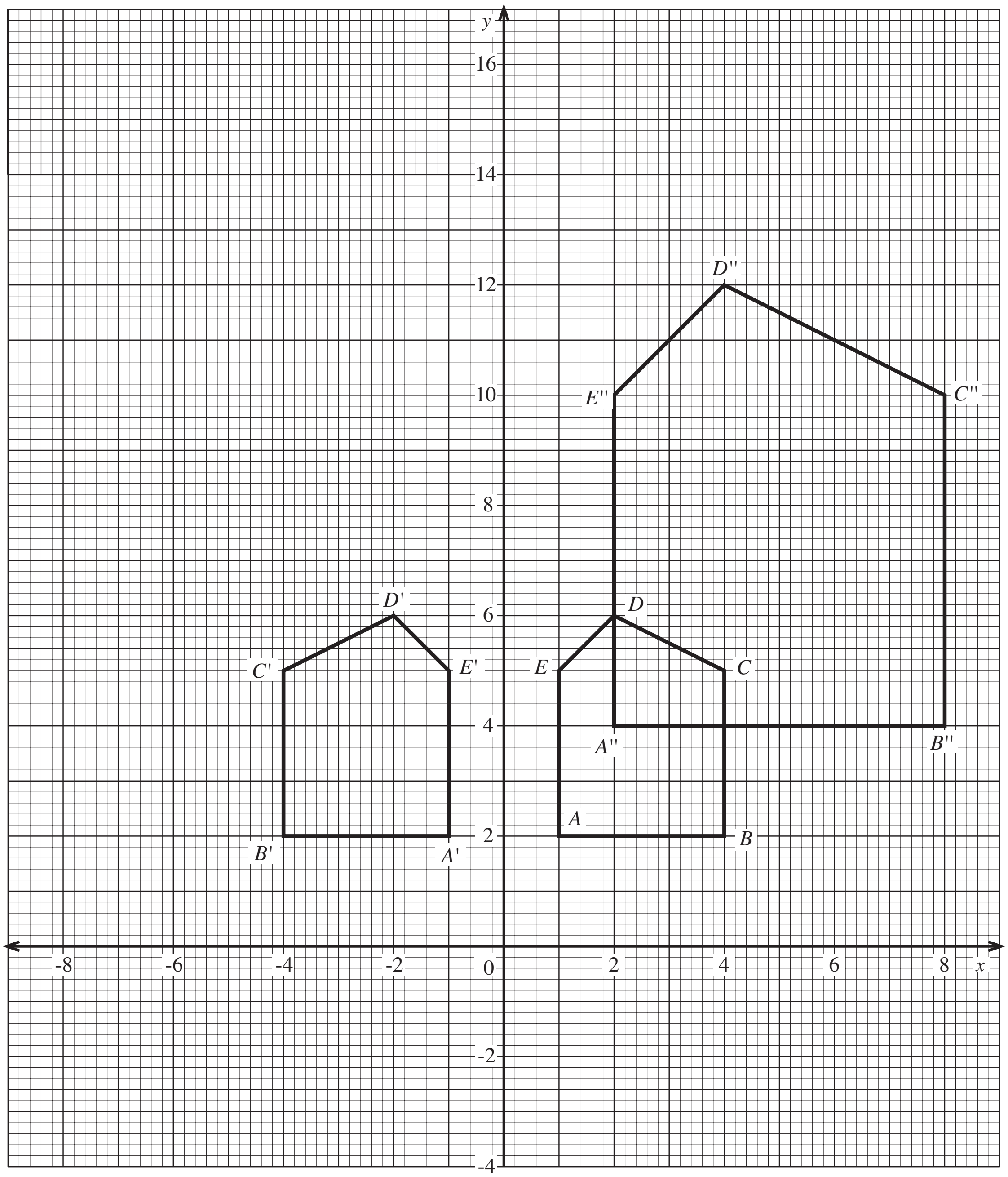
(c) (i) Rotation, about origin, of $90°$, anticlockwise.

(ii) Enlargement with scale factor 2.

(d) Shown on graph (see next page).

(e) Shown on graph (see next page).

Solutions for Worked Example 3 (d) and (e).



Worked Example 4

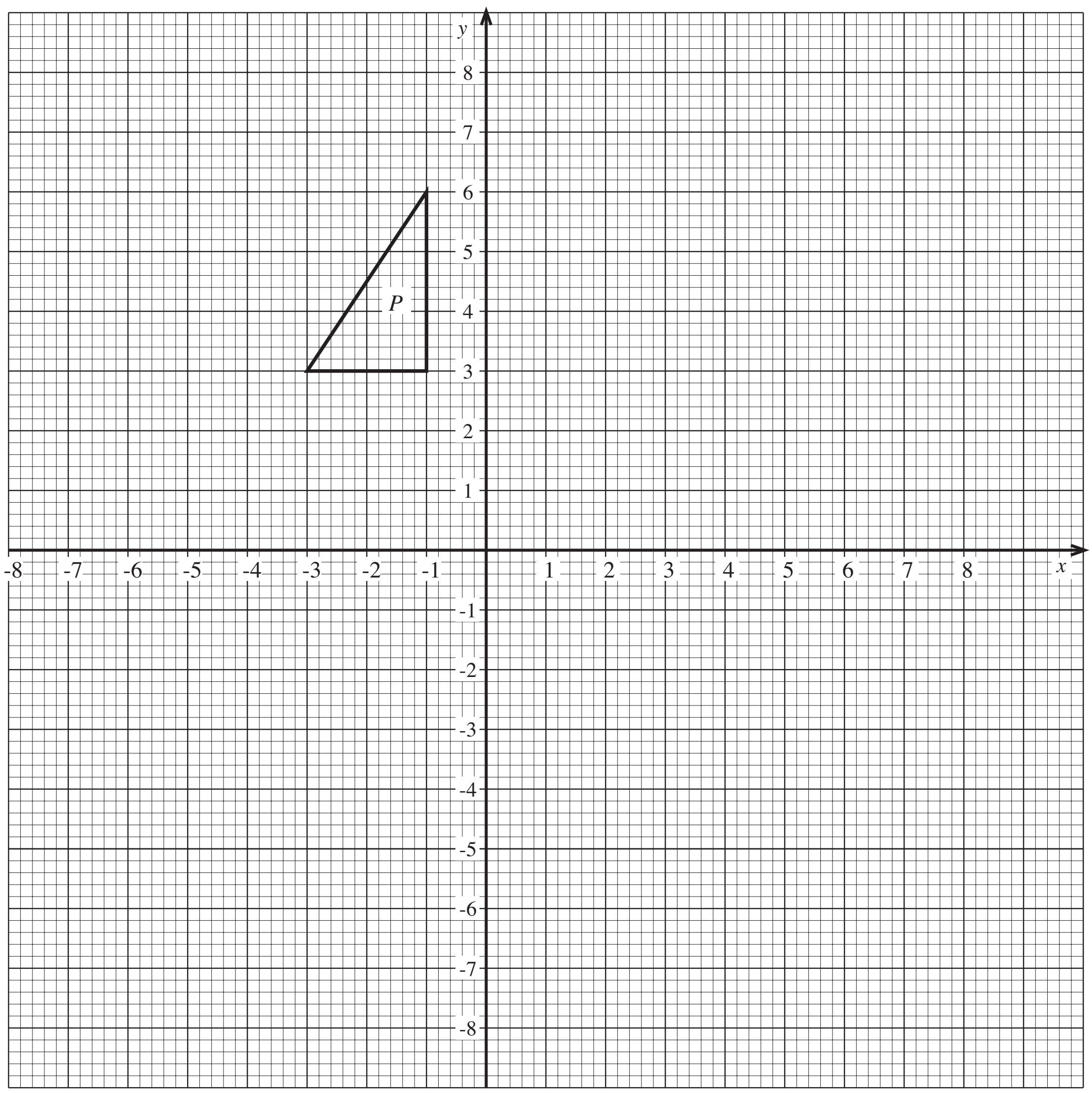
(a) Using the graph paper below, perform the following transformations:

(i) Reflect triangle P in the y-axis.
Label its image Q.

(ii) Draw the line $y = x$ and reflect triangle Q in this line.
Label its image R.

(iii) Describe, in words, the single geometric transformation which maps triangle P onto triangle R.

(iv) Reflect triangle Q in the x-axis.
Label its image S.



(v) Write down the 2×2 matrix for the transformation which maps triangle P onto triangle S.

(b) (i) Write down the 2×2 matrices for

a) a reflection in the y-axis

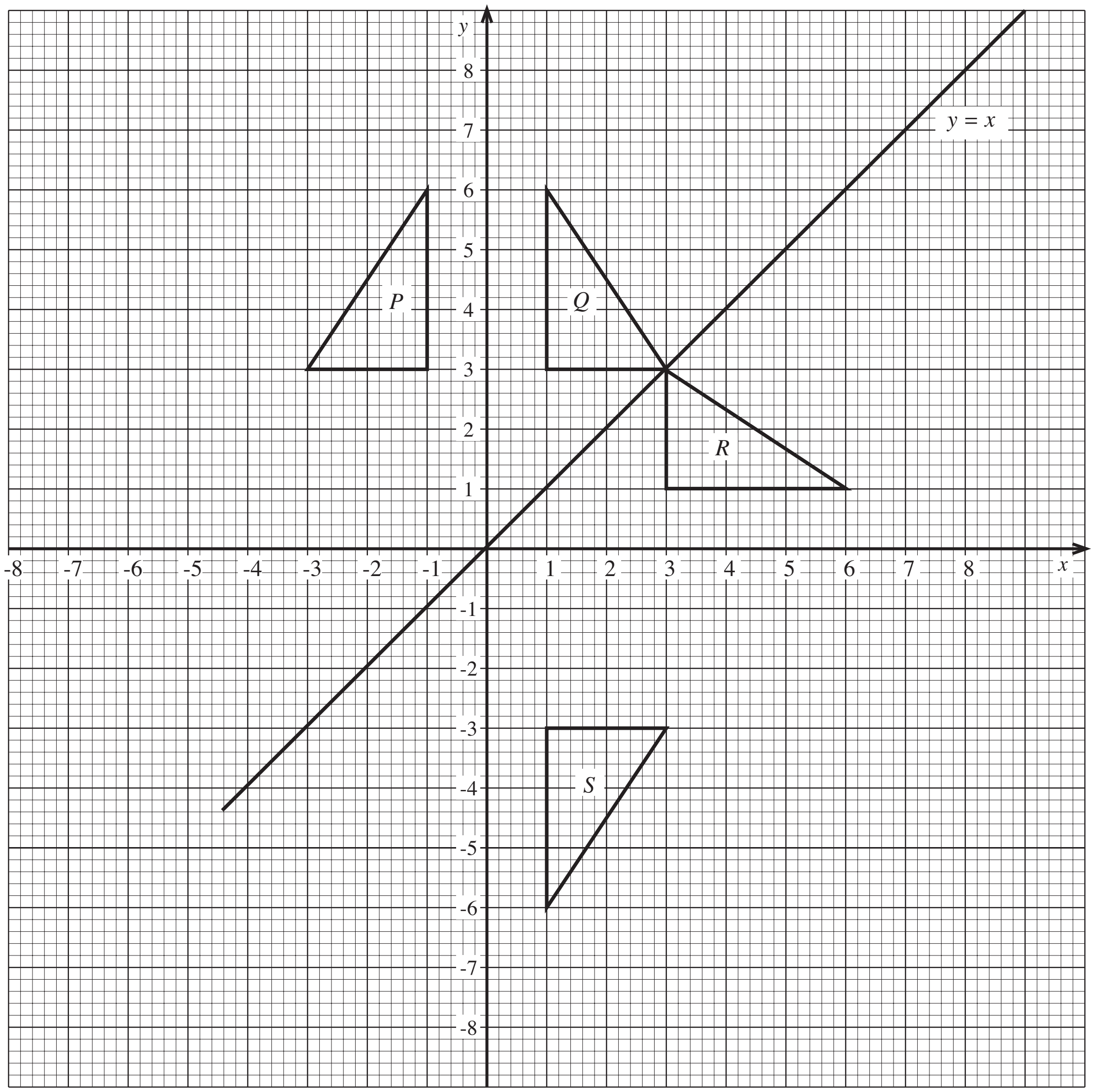
b) a reflection in the line $y = x$.

(ii) Using the two matrices in (b) (i) above, obtain a SINGLE matrix for a reflection in the y-axis followed by a reflection in the line $y = x$.

(CXC)

Solution

(a) (i) and (ii) See diagram below.



(iii) Rotation about origin of $90°$ in clockwise direction

(iv) See diagram above.

(v) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) (i) a) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

(ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Exercises

1. (a) Write down the 2×2 matrix, R, which represents a reflection in the y-axis.

   (b) Write down the 2×2 matrix, N, which represents a clockwise rotation of $180°$ about the origin.

   (c) Write down the 2×1 matrix, T, which represents a translation of -3 units parallel to the x-axis and 5 units parallel to the y-axis.

   (d) The point P (6, 11) undergoes the following combined transformations such that

   $RN(P)$ maps P to P'

   $T(P)$ maps P to P''

   Determine the coordinates of P' and P''.

(CXC)

2. The vertices of triangle ABC have coordinates A (1, 1), B (2, 1) and C (1, 2). Matrix T transforms triangle ABC into triangle $A'B'C'$. The coordinates of the vertices of triangle $A'B'C'$ are A' (3, 3), B' (6, 3) and C' (3, 6).

   (a) Express the transformation matrix, T, in the form $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$.

   (b) Write a complete geometrical description of the transformation T.

   (c) State the ratio of the areas of triangle ABC to $A'B'C'$.

(CXC)

3. (a) Using a scale of 1 cm to represent 1 unit on BOTH the x and the y-axes, draw on graph paper the triangle PQR and $P'Q'R'$ such that $P(-3, -2)$, $Q(-2, -2)$, $R(-2, -4)$ and $P'(6, 4)$, $Q'(4, 4)$ and $R'(4, 8)$.

   (b) Describe FULLY the transformation, G, which maps triangle PQR onto triangle $P'Q'R'$.

(c) The transformation, M, is a reflection in the line $y = -x$.

On the same diagram, draw and label the triangle $P''Q''R''$, the image of triangle $P'Q'R'$ under the transformation M.

(d) Write down the 2×2 matrix for

(i) transformation G

(ii) transformation M

(iii) transformation G followed by M.

(CXC)

4. The matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(a) Give a geometrical interpretation of the transformation represented by R.

(b) Find R^{-1}.

(c) Give a geometrical interpretation of the transformation represented by R^{-1}.

5. The transformations L, M and N are defined by

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(a) Explain the geometrical effect of the transformation M followed by L.

(b) Show that $N = LM$.

6. $R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(a) Find R^2.

(b) Find RS.

(c) Describe the geometrical transformation represented by RS.

7. The matrix R is given by $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

(a) Find R^2.

(b) Describe the geometrical transformation represented by R^2.

(c) Describe the geometrical transformation represented by R.